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Robust Control of Linear Parametrically Varying Systems with Bounded Rates

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An approach to the control of linear parametrically varying systems that takes the rate of parameter variations into account and also guarantees robustness against parametric and dynamic uncertainties is discussed. To illustrate the technique we consider a missile control problem that has been extensively studied in the literature. Nonlinear simulations show that the proposed method is successful in achieving desired performance and robustness goals.

Nomenclature

d	= diameter, 0.75 ft
F_z	= $0.7 p_0 C_n(\alpha, \delta, M) M^2 S$, lb
I_y	= pitch moment of inertia, 182.5 slug-ft^2
L_2	= space of vector-valued signals of finite energy defined on $[0, \infty)$
M	= Mach number of the missile, 2–4
M_y	= $0.7 p_0 C_m(\alpha, \delta, M) M^2 S d$, ft-lb
m	= mass of missile, 13.98 slug
n_z	= normal acceleration of the missile (per g)
p_0	= static pressure at 20,000 ft, 973.3 lb/ft^2
q	= pitch rate, rad/s
S	= reference area, 0.44 ft^2
sy(M)	= $M + M^T$
u	= speed along missile centerline, $V \cos(\alpha)$ ft/s
V	= velocity of the missile, Mss, ft/s
α	= angle of attack, rad
δ	= tail fin deflection, rad

I. Introduction

THE classical approach to gain-scheduling control of a nonlinear system is to suitably parameterize the desired set of operating conditions, linearize the system around each operating point, design a linear controller at a finite number of these operating points, and interpolate the resulting family of controllers to obtain a controller for the original nonlinear system.

In an actual implementation of such a controller, the operating condition will vary with time. However, in the described procedure it is difficult to take these time variations into account to provide a priori guarantees for stability or good performance of the closed-loop system.

For a linear system that depends on some parameters, a so-called linear parametrically varying (LPV) system, it has been illustrated in Ref. 1 that controllers designed for frozen parameters might fail to be stabilizing if these parameters are varying in time. Hence, it is necessary to rely on analysis techniques for time-varying systems to provide stability and performance guarantees if designing a controller for LPV systems with time-varying parameters.

On the basis of results from robust control for linear time invariant systems, a systematic procedure to design parameter dependent controllers for LPV systems has been proposed in Refs. 2–5. These algorithms allow for direct calculation of the controller after solving scaled linear matrix inequalities (LMIs) without any need for an ad hoc interpolation technique. However, because the techniques of Refs. 2–5 provide stability and performance guarantees with a parameter independent Lyapunov function, one cannot include a priori bounds on the rate of variation of the time-varying parameters that introduce conservatism.

Based on analysis results for time-varying systems[6] we will employ in this paper parameter-dependent Lyapunov functions to explicitly take bounds on the rate of change of the parameters into account.[7–12] In addition, we extend this scenario by also incorporating robustness guarantees against possibly structured uncertainties in the design procedure. In contrast to previous work,[6,9,13] we base our arguments on the framework of integral quadratic constraints.[14,15] In this technique, the properties of the uncertainties are described through certain energy relations that increase the flexibility of the possible types of uncertainties that can be incorporated.

The paper is structured as follows. We first describe the underlying uncertain LPV system. Then we show how we can incorporate structural information about the uncertainties by using a suitable class of scalings. This leads to the analysis characterization for robust stability and performance of the uncertain LPV system in terms of scaled differential LMIs in the Lyapunov function and in the possibly parameter-dependent scalings. In the corresponding LPV synthesis problem, the to-be-designed controller parameters enter these inequalities in a nonlinear fashion. We reveal how to transform the controller parameters into new variables that enter the inequalities affinely.[6,7,16,17] Finally, we discuss how to arrive at a computationally tractable design procedure based on the solution of a finite number of linear matrix inequalities.[18]

The theory is illustrated on a missile benchmark problem, as studied in Refs. 12 and 19. In contrast to Ref. 12, where only the nominal performance LPV problem is considered, we also address robust performance issues. Starting from a nonlinear model, we derive an uncertain LPV system[4,20] and translate the design specifications into suitable weighting functions. The resulting interconnection structure forms the basis of our design procedure that requires solving scaled linear matrix inequalities with a D/K-like iteration similar to that in μ-synthesis.[21] Finally, we validate the resulting robust parameter-dependent controller by nonlinear simulations.

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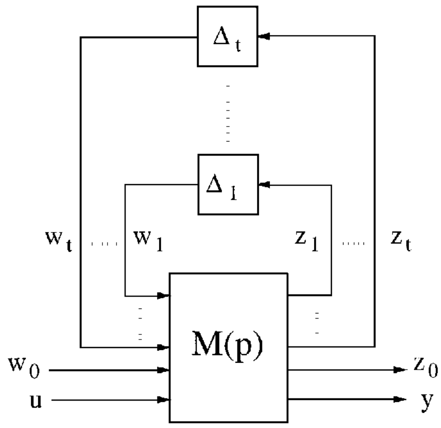

Fig. 1 LPV system with uncertainty and performance channels.

II. Theory of Robust LPV Design

A. Description of Uncertain LPV Systems

We call

$$\dot{\boldsymbol{x}}(t) = A(\boldsymbol{p}(t))\boldsymbol{x}(t) + G(\boldsymbol{p}(t))\boldsymbol{w}(t) + B(\boldsymbol{p}(t))\boldsymbol{u}(t)$$
$$z(t) = H(\boldsymbol{p}(t))\boldsymbol{x}(t) + F(\boldsymbol{p}(t))\boldsymbol{w}(t) + E(\boldsymbol{p}(t))\boldsymbol{u}(t) \qquad (1)$$
$$\boldsymbol{y}(t) = C(\boldsymbol{p}(t))\boldsymbol{x}(t) + D(\boldsymbol{p}(t))\boldsymbol{w}(t)$$

an LPV system, a linear system whose describing matrices depend, possibly in a nonlinear fashion, on a time-varying parameter vector $\boldsymbol{p}(t)$. The parameter vector $\boldsymbol{p}(t)$ and its derivative $\dot{\boldsymbol{p}}(t)$ are assumed to be contained in the a priori given compact sets P and P_r, respectively, and $A(\boldsymbol{p})$, $G(\boldsymbol{p})$, $B(\boldsymbol{p})$, $H(\boldsymbol{p})$, $F(\boldsymbol{p})$, $E(\boldsymbol{p})$, $C(\boldsymbol{p})$, and $D(\boldsymbol{p})$ are continuous functions defined on P. Moreover, $\boldsymbol{x}(t)$, $\boldsymbol{u}(t)$, and $\boldsymbol{y}(t)$ denote the state, the control input, and the measured output signals, respectively. In addition, we assume that the system is affected by structured uncertainties. For that purpose, the signals $\boldsymbol{w}(t)$ and $\boldsymbol{z}(t)$ are partitioned as $[w_0(t)^T, w_1(t)^T, \ldots, w_k(t)^T]^T$ and $[z_0(t)^T, z_1(t)^T, \ldots, z_k(t)^T]^T$. In this partition, $w_0 \to z_0$ defines the performance channel and the uncertainties that are viewed as operators $\Delta_i : L_2 \to L_2$ that enter as

$$w_i(t) = (\Delta_i z_i)(t), \qquad i = 1, \ldots, k \qquad (2)$$

In LPV controller design, the parameter $\boldsymbol{p}(t)$ is assumed to be online measurable. Hence, we assume that the controller also admits an LPV structure and can be described as

$$\dot{\boldsymbol{x}}_c(t) = A_c(\boldsymbol{p}(t))\boldsymbol{x}_c(t) + B_c(\boldsymbol{p}(t))\boldsymbol{y}(t)$$
$$\boldsymbol{u}(t) = C_c(\boldsymbol{p}(t))\boldsymbol{x}_c(t) + D_c(\boldsymbol{p}(t))\boldsymbol{y}(t) \qquad (3)$$

where $A_c(\boldsymbol{p})$, $B_c(\boldsymbol{p})$, $C_c(\boldsymbol{p})$, and $D_c(\boldsymbol{p})$ are the to-be-designed continuous functions on P. The resulting controlled system (Fig. 1) is then given by

$$\dot{\boldsymbol{\xi}}(t) = \mathcal{A}(\boldsymbol{p}(t))\boldsymbol{\xi}(t) + \mathcal{B}(\boldsymbol{p}(t))\boldsymbol{w}(t)$$
$$z(t) = \mathcal{C}(\boldsymbol{p}(t))\boldsymbol{\xi}(t) + \mathcal{D}(\boldsymbol{p}(t))\boldsymbol{w}(t) \qquad (4)$$

with state $\boldsymbol{\xi}(t) = [\boldsymbol{x}(t)^T \;\; \boldsymbol{x}_c(t)^T]^T$, with the uncertainties entering as Eq. (2), and with

B. Analysis of Uncertain LPV Systems

In this section we provide an analysis result that guarantees robust stability and robust performance for the uncertain LPV system described by Eqs. (4) and (2). We follow Ref. 14 and assume that we can specify a set of symmetric matrices

$$\begin{bmatrix} Q_i & S_i^T \\ S_i & R_i \end{bmatrix}$$

called scalings, such that the uncertainty Δ_i satisfies the integral quadratic constraint (IQC)

$$\int_0^t \begin{bmatrix} (\Delta_i z_i)(t) \\ z_i(t) \end{bmatrix}^T \begin{bmatrix} Q_i & S_i^T \\ S_i & R_i \end{bmatrix} \begin{bmatrix} (\Delta_i z_i)(t) \\ z_i(t) \end{bmatrix} \mathrm{d}t \geq 0 \qquad (5)$$

for all $t \geq 0$ and all $z_i \in L_2$. The class of the employed scalings captures information about the properties and about the size of the uncertainty.

Out of a multitude of possibilities, we only mention two typical examples.

The first is time-varying parametric uncertainty: Let Δ_i be given by $(\Delta_i z_i)(t) := \delta_i(t) z_i(t)$ with an arbitrary real time-varying parameter bounded as $|\delta_i(t)| \leq 1$. Then the inequality (5) holds for all blocks satisfying

$$Q_i < 0, \qquad R_i = -Q_i, \qquad S_i + S_i^T = 0$$

The second example is general dynamic uncertainty: Let $\Delta_i : L_2 \to L_2$ be a causal operator whose L_2-gain is bounded by one. Then Eq. (5) is satisfied for all blocks with

$$Q_i = q_i I < 0, \qquad R_i = -Q_i, \qquad S_i = 0$$

As a performance specification we require the L_2-gain of the channel $w_0 \to z_0$ to be bounded by γ, i.e., $\|z_0\|_2 \leq \gamma \|w_0\|_2$. Clearly, this condition can be written in the form (5) with the fixed scalings

$$Q_0 = -\gamma I, \qquad R_0 = (1/\gamma) I, \qquad S_0 = 0$$

Finally, all of the individual scalings are collected in block-diagonal matrices $Q = \mathrm{diag}(Q_0, \ldots, Q_k)$, $R = \mathrm{diag}(R_0, \ldots, R_k)$, and $S = \mathrm{diag}(S_0, \ldots, S_k)$ in the corresponding classes $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$.

For any matrix-valued continuously differentiable function $f(\boldsymbol{p})$ defined on P, let us finally introduce the abbreviation

$$f'(\boldsymbol{p}, \dot{\boldsymbol{p}}) = \sum_{j=1}^{m} \frac{\partial f}{\partial \boldsymbol{p}_i}(\boldsymbol{p}) \dot{\boldsymbol{p}}_i$$

to be viewed as a function on $P \times P_r$. Note that this definition is simply motivated by the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{p}(t)) = f'(\boldsymbol{p}(t), \dot{\boldsymbol{p}}(t))$$

After these preparations, we can formulate the central analysis result that guarantees robust stability and robust performance for the uncertain LPV system (4) and (2) in terms of the solvability of a scaled differential linear matrix inequality.

Theorem: Suppose there exist continuously differentiable functions $\mathcal{X}(\boldsymbol{p})$, $Q(\boldsymbol{p}) \in \mathcal{Q}$, $R(\boldsymbol{p}) \in \mathcal{R}$, $S(\boldsymbol{p}) \in \mathcal{S}$ defined on P that satisfy the inequalities

$$\begin{bmatrix} \mathcal{A}(\boldsymbol{p}) & \mathcal{B}(\boldsymbol{p}) \\ \mathcal{C}(\boldsymbol{p}) & \mathcal{D}(\boldsymbol{p}) \end{bmatrix} = \left[\begin{array}{cc|c} A(\boldsymbol{p}) + B(\boldsymbol{p})D_c(\boldsymbol{p})C(\boldsymbol{p}) & B(\boldsymbol{p})C_c(\boldsymbol{p}) & G(\boldsymbol{p}) + B(\boldsymbol{p})D_c(\boldsymbol{p})D(\boldsymbol{p}) \\ B_c(\boldsymbol{p})C(\boldsymbol{p}) & A_c(\boldsymbol{p}) & B_c(\boldsymbol{p})D(\boldsymbol{p}) \\ \hline H(\boldsymbol{p}) + E(\boldsymbol{p})D_c(\boldsymbol{p})C(\boldsymbol{p}) & E(\boldsymbol{p})C_c(\boldsymbol{p}) & F(\boldsymbol{p}) + E(\boldsymbol{p})D_c(\boldsymbol{p})D(\boldsymbol{p}) \end{array}\right]$$

$$\mathcal{X}(\boldsymbol{p}) > 0$$

$$\begin{bmatrix} \mathrm{sy}(\mathcal{X}(\boldsymbol{p})\mathcal{A}(\boldsymbol{p})) + \mathcal{X}'(\boldsymbol{p},\dot{\boldsymbol{p}}) & \mathcal{X}(\boldsymbol{p})\mathcal{B}(\boldsymbol{p}) + (S(\boldsymbol{p})\mathcal{C}(\boldsymbol{p}))^T & \mathcal{C}(\boldsymbol{p})^T R(\boldsymbol{p}) \\ \mathcal{B}^T(\boldsymbol{p})\mathcal{X}(\boldsymbol{p}) + S(\boldsymbol{p})\mathcal{C}(\boldsymbol{p}) & \mathrm{sy}(S(\boldsymbol{p})\mathcal{D}(\boldsymbol{p})) + Q(\boldsymbol{p}) & \mathcal{D}(\boldsymbol{p})^T R(\boldsymbol{p}) \\ R(\boldsymbol{p})\mathcal{C}(\boldsymbol{p}) & R(\boldsymbol{p})\mathcal{D}(\boldsymbol{p}) & -R(\boldsymbol{p}) \end{bmatrix} < 0 \tag{6}$$

for all $\boldsymbol{p} \in P$ and all $\dot{\boldsymbol{p}} \in P_r$.

Then, for any parameter curve $\boldsymbol{p}(t) \in P$ with $\dot{\boldsymbol{p}}(t) \in P_r$ and for all uncertainties (2) satisfying the IQC (5), the uncertain LPV system (4) and (2) remains stable, and the L_2-gain of the performance channel $w_0 \to z_0$ is bounded by γ.

The simple proof can be provided by Lyapunov arguments as found, for example, in Refs. 4, 6, 11, and 15.

As an essential point, the desired stability and performance properties along parameter curves are implied by the validity of the inequalities (6) on the parameter set $P \times P_r$. Note that this result admits a multitude of specializations. In particular, one could assume $\mathcal{X}$ and the scalings Q, R, and S to be independent of the parameter.[2,3,5] Because $\mathcal{X}' = 0$, we observe that the inequalities do not depend on $\dot{\boldsymbol{p}} \in P_r$ such that robust stability and performance is guaranteed for arbitrarily fast parameter variations.

In a concrete application, we have to numerically search for a parameter-dependent Lyapunov matrix $\mathcal{X}(\boldsymbol{p})$ and parameter-dependent scalings $Q(\boldsymbol{p})$, $R(\boldsymbol{p})$, and $S(\boldsymbol{p})$ that render the differential LMIs (6) satisfied. For that purpose we choose continuously differentiable basis functions $f_1(\boldsymbol{p}), \ldots, f_l(\boldsymbol{p})$ and search for the coefficient matrices $\mathcal{X}_j$, Q_j, R_j and S_j in the expansion

$$[\mathcal{X}(\boldsymbol{p}) \quad Q(\boldsymbol{p}) \quad R(\boldsymbol{p}) \quad S(\boldsymbol{p})] = \sum_{j=1}^{l} f_j(\boldsymbol{p})[\mathcal{X}_j \quad Q_j \quad R_j \quad S_j]$$

The resulting infinite number of LMIs parametrized by $(\boldsymbol{p},\dot{\boldsymbol{p}}) \in P \times P_r$ are reduced to finitely many inequalities by picking a finite number of points in P and P_r. If P_r is described as a convex combination of finitely many vertices, it suffices to choose the extreme points because parameter $\dot{\boldsymbol{p}}$ appears linearly in Eq. (6).

C. Controller Synthesis

The synthesis problem consists of designing a controller as defined by Eq. (3) that minimizes the robust performance level γ as characterized in theorem. We observe that the unknown functions, the Lyapunov matrix, the scalings, and the controller parameters enter the inequalities (6) in a nonlinear fashion. It has been shown quite recently[6,16,17] how the synthesis inequalities can be linearized, for fixed scalings, through a suitable nonlinear transformation

$$\left[\mathcal{X}(\boldsymbol{p}) \begin{bmatrix} A_c(\boldsymbol{p}) & B_c(\boldsymbol{p}) \\ C_c(\boldsymbol{p}) & D_c(\boldsymbol{p}) \end{bmatrix}\right] \to \left[X(\boldsymbol{p}) \;\; Y(\boldsymbol{p}) \begin{bmatrix} K(\boldsymbol{p},\dot{\boldsymbol{p}}) & L(\boldsymbol{p}) \\ M(\boldsymbol{p}) & N(\boldsymbol{p}) \end{bmatrix}\right]$$

of the Lyapunov matrix $\mathcal{X}(\boldsymbol{p})$ and of the controller parameters $A_c(\boldsymbol{p})$, $B_c(\boldsymbol{p})$, $C_c(\boldsymbol{p})$, and $D_c(\boldsymbol{p})$ into the new unknowns $X(\boldsymbol{p})$, $Y(\boldsymbol{p})$, $K(\boldsymbol{p})$, $L(\boldsymbol{p})$, $M(\boldsymbol{p})$, $N(\boldsymbol{p})$. Let us partition $\mathcal{X}(\boldsymbol{p}) > 0$ and its inverse $\mathcal{X}(\boldsymbol{p})^{-1} > 0$ as

$$\mathcal{X}(\boldsymbol{p}) = \begin{bmatrix} X(\boldsymbol{p}) & U(\boldsymbol{p}) \\ U^T(\boldsymbol{p}) & \tilde{X}(\boldsymbol{p}) \end{bmatrix}, \qquad \mathcal{X}(\boldsymbol{p})^{-1} = \begin{bmatrix} Y(\boldsymbol{p}) & V(\boldsymbol{p}) \\ V^T(\boldsymbol{p}) & \tilde{Y}(\boldsymbol{p}) \end{bmatrix}$$

Note that the blocks are related as

$$Y(\boldsymbol{p})X(\boldsymbol{p}) + V(\boldsymbol{p})U^T(\boldsymbol{p}) = I \tag{7}$$

and $U^T(\boldsymbol{p})Y(\boldsymbol{p}) + \tilde{X}(\boldsymbol{p})V^T(\boldsymbol{p}) = 0$. The desired transformation is now given by

$$\begin{aligned} K(\boldsymbol{p},\dot{\boldsymbol{p}}) &= X'(\boldsymbol{p},\dot{\boldsymbol{p}})Y(\boldsymbol{p}) + U'(\boldsymbol{p},\dot{\boldsymbol{p}})V^T(\boldsymbol{p}) \\ &\quad + X(\boldsymbol{p})[A(\boldsymbol{p}) + B(\boldsymbol{p})D_c(\boldsymbol{p})C(\boldsymbol{p})]Y(\boldsymbol{p}) \\ &\quad + U(\boldsymbol{p})A_c(\boldsymbol{p})V^T(\boldsymbol{p}) + U(\boldsymbol{p})B_c(\boldsymbol{p})C(\boldsymbol{p})Y(\boldsymbol{p}) \\ &\quad + X(\boldsymbol{p})B(\boldsymbol{p})C_c(\boldsymbol{p})V^T(\boldsymbol{p}) \\ L(\boldsymbol{p}) &= X(\boldsymbol{p})B(\boldsymbol{p})D_c(\boldsymbol{p}) + U(\boldsymbol{p})B_c(\boldsymbol{p}) \\ M(\boldsymbol{p}) &= D_c(\boldsymbol{p})C(\boldsymbol{p})Y(\boldsymbol{p}) + C_c(\boldsymbol{p})V^T(\boldsymbol{p}), \qquad N(\boldsymbol{p}) = D_c(\boldsymbol{p}) \end{aligned} \tag{8}$$

This transformation is inverted as follows. Let $X(\boldsymbol{p}) > 0$, $Y(\boldsymbol{p}) > 0$, $K(\boldsymbol{p},\dot{\boldsymbol{p}})$, $L(\boldsymbol{p})$, $M(\boldsymbol{p})$, and $N(\boldsymbol{p})$ be given. Because $X(\boldsymbol{p})$ and $Y(\boldsymbol{p})$ are positive definite, one can easily find continuously differentiable nonsingular matrix functions $U(\boldsymbol{p})$ and $V(\boldsymbol{p})$ that satisfy Eq. (7). Then one computes the functions $A_c(\boldsymbol{p},\dot{\boldsymbol{p}})$, $B_c(\boldsymbol{p})$, $C_c(\boldsymbol{p})$, and $D_c(\boldsymbol{p})$ by solving Eq. (8), and the still undefined block $\tilde{X}(\boldsymbol{p})$ of $\mathcal{X}(\boldsymbol{p})$ is obtained by solving $U^T(\boldsymbol{p})Y(\boldsymbol{p}) + \tilde{X}(\boldsymbol{p})V^T(\boldsymbol{p}) = 0$.

The analysis inequalities (6) are transformed into the corresponding synthesis inequalities by performing the substitutions

$$\mathcal{X} \to \begin{bmatrix} X & I \\ I & Y \end{bmatrix}, \qquad \mathcal{X}' \to \begin{bmatrix} X' & 0 \\ 0 & -Y' \end{bmatrix}$$

$$\begin{bmatrix} \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \to \left[\begin{array}{cc|c} XA + LC & K & XG + LD \\ A + BNC & AY + BM & G + BND \\ \hline H + ENC & HY + EM & F + END \end{array}\right]$$

where we omitted the dependence on $\boldsymbol{p}$ and $\dot{\boldsymbol{p}}$ for notational simplicity.

The resulting inequalities are still not linear in the new variables and the scalings together. Therefore, we have to resort to a D/K-like iteration scheme that proceeds as follows: Start with fixed constant scalings $Q \in \mathcal{Q}$, $R \in \mathcal{R}$, $S \in \mathcal{S}$ and iterate the following two steps until the performance level γ cannot be improved.

1) For fixed scalings, minimize γ over $X(\boldsymbol{p})$, $Y(\boldsymbol{p})$ and the transformed controller parameters $K(\boldsymbol{p},\dot{\boldsymbol{p}})$, $L(\boldsymbol{p})$, $M(\boldsymbol{p})$, $N(\boldsymbol{p})$ satisfying the synthesis inequalities.

2) For fixed $K(\boldsymbol{p},\dot{\boldsymbol{p}})$, $L(\boldsymbol{p})$, $M(\boldsymbol{p})$, and $N(\boldsymbol{p})$, compute $\mathcal{A}(\boldsymbol{p})$, $\mathcal{B}(\boldsymbol{p})$, $\mathcal{C}(\boldsymbol{p})$, and $\mathcal{D}(\boldsymbol{p})$, and minimize the performance level γ over $\mathcal{X}(\boldsymbol{p})$ and the scalings $Q(\boldsymbol{p}) \in \mathcal{Q}$, $R(\boldsymbol{p}) \in \mathcal{R}$, $S(\boldsymbol{p}) \in \mathcal{S}$ satisfying the analysis inequalities (6).

As for analysis, the numerical solution of the optimization problem in the first step proceeds via choosing suitable basis function and gridding the parameter set. The second step consists of solving an analysis problem as described in the preceding section. Hence, both problems amount to solving a finite number of standard linear matrix inequalities. The iteration is stopped if the performance level γ cannot be significantly decreased by any of the two optimization problems.

In this procedure, the controller matrix $A_c(\boldsymbol{p},\dot{\boldsymbol{p}})$ depends as well on the variable $\dot{\boldsymbol{p}}$. To implement this controller, one requires not only to measure the parameter value $\boldsymbol{p}(t)$ itself but also its rate of change $\dot{\boldsymbol{p}}(t)$. To avoid this undesired structure, one can proceed as follows. In the numerical scheme, we let $K(\boldsymbol{p})$ only depend on the variable $\boldsymbol{p}$ and we let either $X(\boldsymbol{p})$ or $Y(\boldsymbol{p})$ be independent of $\boldsymbol{p}$ (Refs. 7 and 8). Exploiting the freedom in the choice of $U(\boldsymbol{p})$ and $V(\boldsymbol{p})$ allows construction of a controller that depends only on $\boldsymbol{p}$.

1) If $X(\boldsymbol{p})$ is parameter dependent and Y is constant, choose $U(\boldsymbol{p}) = I - X(\boldsymbol{p})Y$ and $V = I$. Differentiating Eq. (7) reveals

$$YX'(\boldsymbol{p},\dot{\boldsymbol{p}}) + VU'(\boldsymbol{p},\dot{\boldsymbol{p}})^T = 0 \tag{9}$$

so that the terms in Eqs. (8) that depend on $\dot{\boldsymbol{p}}$ indeed drop out.

2) If X is constant and $Y(\boldsymbol{p})$ is parameter dependent, choose $U = I$ and $V(\boldsymbol{p}) = I - Y(\boldsymbol{p})X$. This implies $X'(\boldsymbol{p},\dot{\boldsymbol{p}}) = 0$, $U'(\boldsymbol{p},\dot{\boldsymbol{p}}) = 0$ such that, again, the variable $\dot{\boldsymbol{p}}$ disappears in Eqs. (8).

Clearly, restricting $K(\boldsymbol{p})$ not to depend on $\dot{\boldsymbol{p}}$ and $X(\boldsymbol{p})$ or $Y(\boldsymbol{p})$ to be constant introduces conservatism, but it has the benefit of a simpler controller implementation.

To speed up the iterations, we finally remark that we performed all computations after eliminating the transformed controller parameters by using the projection lemma,[3] and by restricting the class of scalings to $\mathcal{S} = \{0\}$.

III. Missile Control Problem

As an application we have chosen a missile benchmark problem that has been extensively studied in Refs. 4, 12, 19, and 20 and is particularly suited for addressing gain scheduling as well as robustness issues. The objective is to design a longitudinal autopilot for a tail-fin controlled missile providing normal acceleration tracking over a large range of speed and angle of attack. To arrive at the design model for analysis and synthesis given in Sec. III.C, the problem specifications given in Sec. III.B are translated into suitable weighting filters and uncertainty models. These are based on the LPV missile model that is described in Sec. III.A.

A. Missile Model

The nonlinear state equations of the missile are

$$\dot{\alpha} = f_1(\alpha, q, \delta, M) = \frac{\cos(\alpha)^2}{mu} F_z(\alpha, \delta, M) + q \tag{10}$$

$$\dot{q} = f_2(\alpha, q, \delta, M) = (M_y/I_y)(\alpha, \delta, M) \tag{11}$$

The aerodynamic nonlinearity and parameter dependence in the missile model are reflected in the normal force and moment coefficients $C_n(\alpha, \delta, M)$ and $C_m(\alpha, \delta, M)$. Taking the missile symmetry into account, it suffices to consider the positive values of the angle of attack. These aerodynamic coefficients are given by the following polynomial expressions:

$$C_n(\alpha, \delta, M) = a_n\alpha^3 + b_n\alpha^2 + c_n[2 + (M/3)]\alpha + d_n\delta$$

$$C_m(\alpha, \delta, M) = a_m\alpha^3 + b_m\alpha^2 - c_m[7 - (8M/3)]\alpha + d_m\delta$$

where the polynomial coefficients are

$$a_n = +0.000103\,\mathrm{deg}^{-3}, \qquad a_m = +0.000215\,\mathrm{deg}^{-3}$$

$$b_n = -0.009450\,\mathrm{deg}^{-2}, \qquad b_m = -0.019500\,\mathrm{deg}^{-2}$$

$$c_n = -0.169600\,\mathrm{deg}^{-1}, \qquad c_m = +0.051000\,\mathrm{deg}^{-1}$$

$$d_n = -0.034000\,\mathrm{deg}^{-1}, \qquad d_m = -0.206000\,\mathrm{deg}^{-1}$$

The coefficients need first to be transformed into radians to comply with the equations of motion (10) and (11). This aerodynamic model is valid for the missile traveling between Mach = 2 and 4 at an altitude of 20,000 ft. Typical maneuvers for this missile result in angle-of-attack values ranging between −20 and +20 deg. Hence, the approximation $\cos(\alpha) \approx 1$ is legitimate. Equation (10) simplifies to

$$\dot{\alpha} = \frac{F_z(\alpha, \delta, M)}{mV} + q \tag{12}$$

One way to obtain an LPV model for the missile is to parameterize the set of all equilibrium models. For any angle of attack $\alpha \in [0, 20]$ (degree) and Mach number $M \in [2, 4]$, the fin deflection and pitch rate

$$\delta(\alpha, M) = -(1/d_m)\left\{a_m\alpha^3 + b_m\alpha^2 - c_m[7 - (8M/3)]\alpha\right\}$$

$$q(\alpha, M) = -\frac{F_z(\alpha, \delta, M)}{mV}$$

lead to an equilibrium of Eqs. (10) and (11). The specific normal force n_z is measured by an accelerometer placed at the center of gravity of the missile. It is defined as $n_z = F_z/W$ where $W = mg$. For convenience we use the notation $K_\alpha = 0.7 p_0 S/m$ at speed of sound at 20,000 ft = (1036.4 ft/s), $K_q = 0.7 p_0 S d/I_y$, $K_n = (\pi/180)(0.7 p_0 S/W)$. The Jacobi linearization of the missile dynamics is then given as

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \hline q \\ n_z \end{bmatrix} = \left[\begin{array}{c|c} A(\alpha, M) & B(\alpha, M) \\ \hline C(\alpha, M) & D(\alpha, M) \end{array}\right] \begin{bmatrix} \alpha \\ q \\ \hline \delta \end{bmatrix}$$

where

$$A = \begin{bmatrix} K_\alpha M \dfrac{\partial C_n}{\partial \alpha} & 1 \\ K_q M^2 \dfrac{\partial C_m}{\partial \alpha} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} K_\alpha M \dfrac{\partial C_n}{\partial \delta} \\ K_q M^2 \dfrac{\partial C_m}{\partial \delta} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \\ K_n M^2 \dfrac{\partial C_n}{\partial \alpha} & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ K_n M^2 \dfrac{\partial C_n}{\partial \alpha} \end{bmatrix}$$

These dynamics represent in all equilibrium points given by $[\delta(\alpha, M), q(\alpha, M)]$ the family of all linearized systems parametrized by $\boldsymbol{p} = (\alpha, M)$. For a particular parameter value $\boldsymbol{p}$ in the allowable parameter set, the LPV dynamics are called frozen and reflect a local linearization of the missile dynamics.

B. Specifications

The specifications to be achieved by the controller have to hold over the whole Mach range $M \in [2, 4]$. Therefore, the system should globally provide normal acceleration command tracking features, with rise time not greater than 0.35 s, overshoot not greater than 10%, and steady-state error not greater than 1%. The measurements available for control are the normal acceleration n_z, the pitch rate q, and the Mach number M. During a typical maneuver the angle of attack ranges within $|\alpha| \leq 20$ deg; the tail-fin deflection rate should not exceed 25 deg/s per commanded g-level.

As very strong simplifications in the missile modeling have been made, we take the robustness issue originating from the uncertainty in the aerodynamic coefficients C_n and C_m into account. The uncertainty levels considered are $\Delta C_n = \pm 10\%$ and $\Delta C_m = \pm 25\%$.

The controller provides fin commands δ_c that are processed through second-order actuator dynamics given by $G_{\mathrm{act}}(s) = \omega_a^2/(s^2 + 2\zeta\omega_a s + \omega_a^2)$, with natural frequency $\omega_a = 150$ rad/s and damping $\zeta = 0.7$.

C. Control Strategy

To realize the specifications over the prescribed operating range, the missile dynamics are reformulated into an uncertain parametrically varying system representation in terms of a linear fractional transformation (LFT). The LPV system with parameter $\boldsymbol{p} = (\alpha, M)$ has an uncertain part arising from the perturbations in the aerodynamic coefficients. We assume that the measurement of the angle of attack is not available. Therefore, we view in our problem the angle of attack in the parameter vector $\boldsymbol{p}$ as uncertain. This strategy has been chosen to illustrate how uncertainties of various origins can be incorporated in the method. This is in contrast to the less conservative approach shown in Ref. 12, where the angle of attack α has been used as schedule measurement during the synthesis and implemented via an observer structure.

The angle of attack α and the uncertainties in the aerodynamic coefficients C_m and C_n are extracted from the system in a linear fractional way and rescaled to $[-1, +1]$. To avoid excitation of unmodeled high-frequency dynamics, a multiplicative input uncertainty Δ_{in} weighted by $W_{\mathrm{in}}(s) = 1.5[(s+2)/(s+80)]$ is placed at the actuator. For the overall missile dynamics including actuator uncertainty, we end up with a block-diagonal uncertainty structure given by $\Delta_u = \mathrm{diag}(\delta_\alpha I_2, \Delta_{C_n}, \Delta_{C_m}, \Delta_{\mathrm{in}})$.

The control architecture for the missile problem is shown in Fig. 2. The tracking specification has been translated into an ideal acceleration model, which the closed-loop system should match. The ideal model comes from Ref. 12 and is $W_{\mathrm{id}}(s) = [144(-0.05s + 1)/(s^2 + 19.2s + 144)]$ for which the allowable error is weighted with $W_{\mathrm{perf}}(s) = [0.5s + 17.321)/(s + 0.0577)]$. The low-frequency gain of W_{perf} is 300 to bound the tracking error to 0.33%. To reflect the tail-fin deflection and deflection rate limits of 20 deg and 25 deg/s per g, respectively, constant weights $W_\delta = 1/20$ and $W_{\dot{\delta}} = 1/25$ have been chosen accordingly. Finally, noise weights $W_{n1} = 0.001$ and $W_{n2} = 0.001$ are used to reflect the measurement imperfections in pitch rate and normal acceleration.

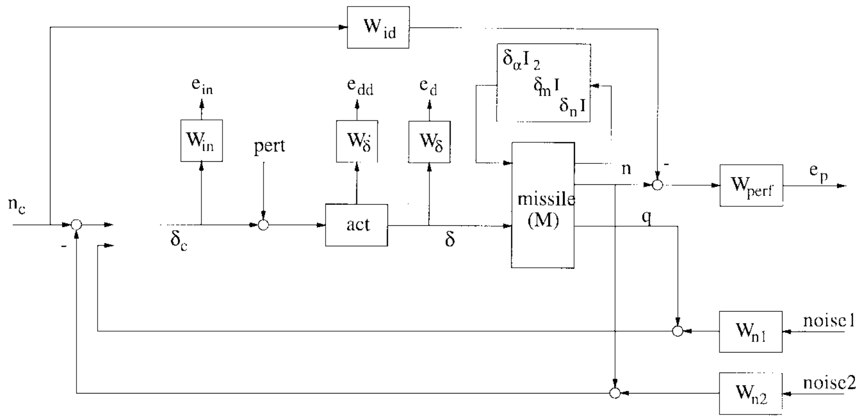

Fig. 2 Controller synthesis interconnection structure.

IV. Results

A. Design of the LPV Controller

In this section we will use the synthesis LMIs in Eq. (6) as derived earlier. Solving the LMIs is done via basis functions and gridding of the parameter space. For the missile control problem, the Mach number M is the remaining parameter for scheduling. The angle of attack α, the aerodynamic coefficients C_n and C_m, and the actuator were all assumed uncertain. Because the Mach number M will be chosen to decrease linearly during the maneuver, we take a basis function as $f_1(M) = M$ (see also Ref. 7). For the function $X(M)$, e.g., thus, we have

$$X(M) = X_0 + X_1 M$$

The other functions depending on the parameter M are $Y(M)$, $Q(M)$, $K(M)$, $L(M)$, $M(M)$, and $N(M)$ and have the same structure as $X(M)$.

The grid of the parameter set $M \in [2, 4]$ consisted of five points equally spaced between Mach = 2 and 4. As the Mach number will decrease from $M = 4$ to $M = 2$ in 5 s in the nonlinear simulations, the parameter rate was taken $|\dot{M}| < 0.5$/s. Further, we use a block diagonal scaling matrix $Q = \text{diag}(Q_0, \ldots, Q_4)$ arising from the uncertainty and the performance channels: the matrix Q_1 of dimension 2×2 for the uncertain α, the two scalar blocks Q_2 and Q_3 for the uncertainty in C_n and C_m, the scalar block Q_4 for the dynamic actuator uncertainty, and the 3×3 block Q_0 for the performance specification. In the first iteration the scalings are set to unity. Once convergence is achieved, the LMIs in Eq. (6) are solved for the last scaling Q that was found in the iteration.

In Sec. II.C we revealed that choosing either X or Y constant leads to a controller that does not require a measurement of the parameter rate. For implementation in the missile control problem, both options were tested. The scheme using X constant and Y parameter dependent converged faster than the one where X was parameter dependent and Y constant. In 14 steps, a γ value of 2.50 was reached for the first scheme, in contrast to the other option, where only a γ value of 3.87 was reached after 20 iteration steps. In Fig. 3 the achieved γ value is set out against the number of iteration steps for both schemes.

The final controller synthesis was carried out on the choice $(X_0, Y(M))$ where $\gamma = 2.5$ was achieved. The scaling matrix Q of the last iteration is used to synthesize the controller using the full synthesis LMIs. The problem size with projected LMIs has 109 decision variables, in contrast to the full LMI problem with 289 decision variables. The complete design cycle takes about 6–8 h on a 200-MHz Pentium-pro computer.

The achieved γ value was 2.51, and the test on a denser parameter grid (with twice the density) gave γ values between 2.22 and 2.45. The achieved performance level given by the proposed scheme is comparable to results given in Ref. 12 ($\gamma = 3.13$) and in Ref. 4 ($\gamma = 3.855$) with the difference that, in our design, parameter rate boundness and uncertainties are taken into account. Viewing the angle of attack as uncertain increases the achievable γ level. Nevertheless, the introduced conservatism is compensated by taking into

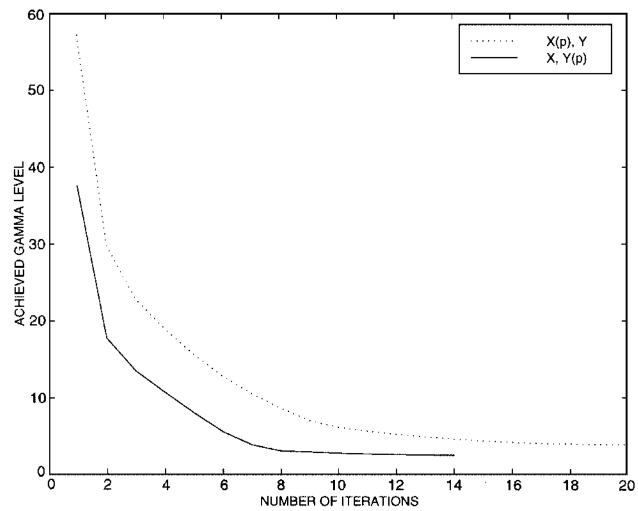

Fig. 3 Achieved performance γ $(X, Y(M))_{14}$ = 2.50 and γ $(X(M), Y)_{20}$ = 3.87 against number of iterations.

account the parameter rate $\dot{M}$. We end up with a lower γ value and a controller implementation with less sensor information than given in Refs. 4 and 12.

Next, we illustrate how the missile problem behaves using different design methods and assumptions on the control and uncertainty structures. Running the same algorithm leaving out the aerodynamic uncertainties provided in 10 iterations an achievable performance of $\gamma = 2.38$ for the case X constant and Y parameter dependent. Two other designs have been conducted using LFT gain scheduling approaches such as given in Refs. 22 and 23 also using the angle of attack α as a scheduling parameter. We achieved γ values of 4.43 and 3.60, respectively. These LFT approaches assume the parameter rates of α and Mach unbounded. It is not surprising that this leads to more conservative γ levels, which correspond to larger overshoots in the normal acceleration step command responses. Finally, to get an indication of the performance level that can be ideally obtained in each operating condition, robust H_∞ controllers have been designed for frozen α and Mach operating points. The resulting L_2-induced norm ranged between $\gamma = 0.3$ in $(\alpha, M) = (20, 4)$ and $\gamma = 0.990$ in $(\alpha, M) = (0, 2)$.

B. Simulations

The nonlinear simulations of the LPV-controlled missile are shown in the Figs. 4–8. The maneuver, during which the Mach number varies from 4 to 2 in 5 s, consists of a series of acceleration step commands, as shown in Fig. 4. The acceleration command response of the LPV-controlled missile has a rise time that is less than the prescribed 0.35 s. The steady-state error is within the required bounds. Overshoot characteristics are also within the limits. Only the step command from 30 to -15 g causes a 3% overshoot violation from the allowed 10%. As a remedy one could try to redesign

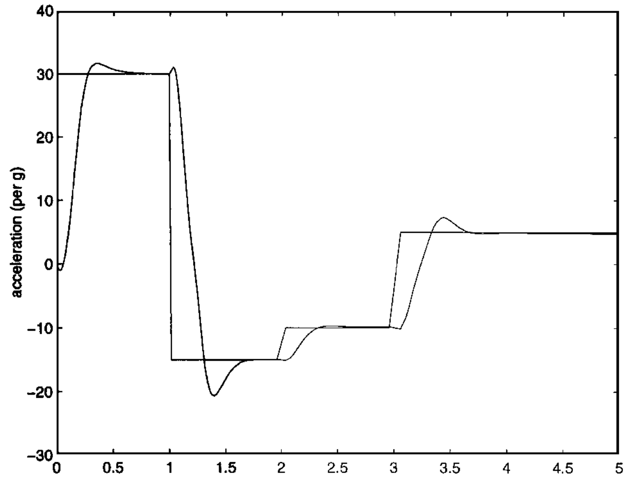

Fig. 4 Normal acceleration n_z for the commanded acceleration scenario n_c of the LPV-controlled missile.

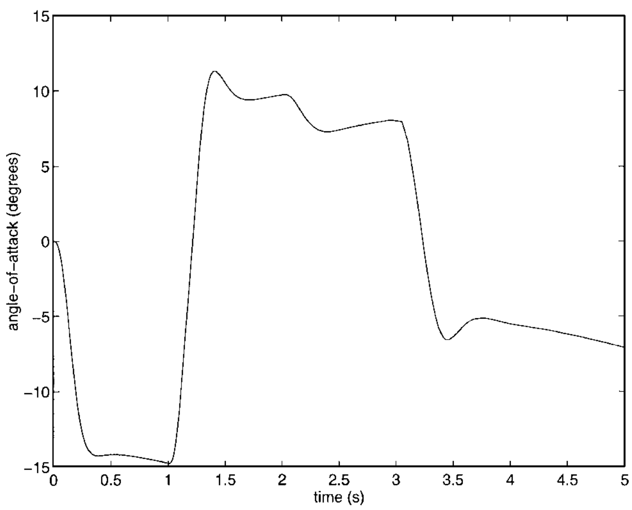

Fig. 5 Angle of attack α for the commanded acceleration scenario n_c of the LPV-controlled missile.

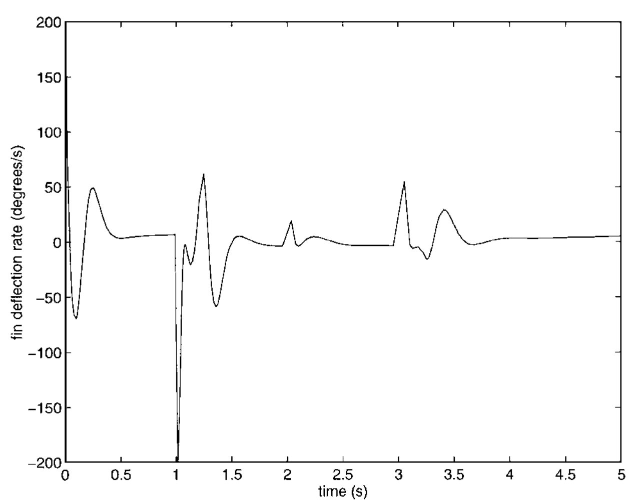

Fig. 6 Fin deflection rate $\dot{\delta}$ for the commanded acceleration scenario n_c of the LPV-controlled missile.

the weightings. A possible choice to enhance damping of the acceleration response is to increase the weight on the fin rate filter $W_{\dot{\delta}}$ inasmuch as maximum fin rate is not reached in the nonlinear simulation. Also the performance filter could be adjusted to further penalize the overshoot (increase high-frequency gain of the filter). However, we left the filters W_{id} and W_{perf}, the same as in Ref. 12, to be able to compare the results. The LPV-controller synthesized here has a larger overshoot, but the responses are faster than the LPV design in Ref. 12. Moreover the LPV controller of Ref. 12 shows angle of attack excursions beyond −20 deg while also commanding four times more fin deflection rate during the −45 g command. Figures 5 and 6 show that in our design the angle-of-attack and the fin deflection rate remain within the prescribed limits. The corresponding fin deflection, which is shown in Fig. 7, is rather small. In that respect the design is conservative. The available control power has not been fully exploited and could be used to enhance the overall system performance. It should be noted that the missile in Ref. 12 runs along a slightly different Mach trajectory. Finally, to demonstrate the robustness properties, Fig. 8 shows the acceleration command responses for all combinations of the aerodynamic uncertainties. As can be seen from Fig. 8, overshoot in the −45 g step is the most sensitive to uncertainties, whereas the other performance characteristics seem to behave well.

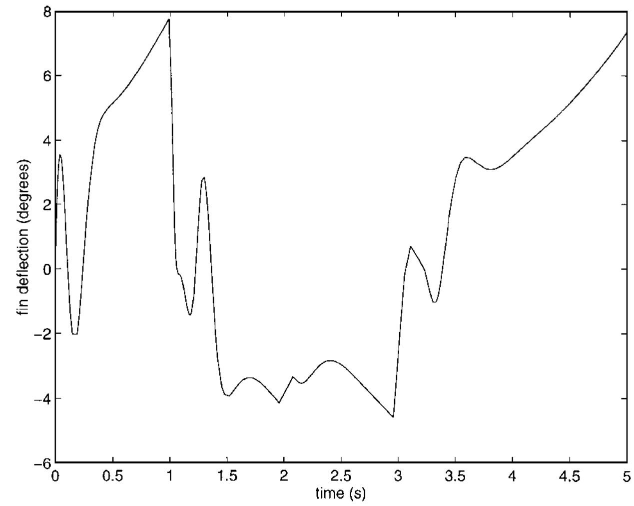

Fig. 7 Fin deflection δ for the commanded acceleration scenario n_c of the LPV-controlled missile.

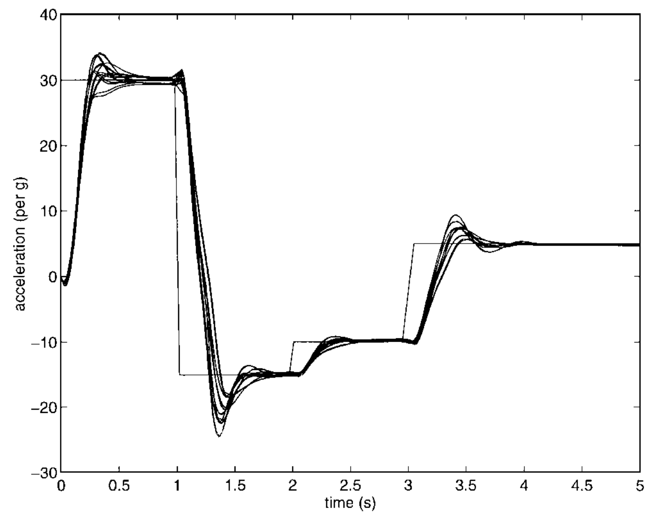

Fig. 8 Command response for all combinations of perturbed aerodynamics.

V. Conclusions

We have developed a controller synthesis technique for linear parameterically varying systems that takes the boundedness of the parameter variation rates into account. This technique gives guaranteed stability and performance levels. Moreover, robustness against uncertainties has been incorporated by the use of scalings. Using basis functions and gridding, the synthesis problem is reduced to an iteration process that involves solving finitely many linear matrix inequalities. The method has been applied to and tested on a missile benchmark problem. Nonlinear simulations reveal that the proposed method is successful in achieving the desired performance and robustness goals. However, there are still many problem areas in the LPV theory that need further research. Especially, the

LPV system modeling approach undertaken is not applicable to general systems. When no analytical expressions for the system dynamics are available, one would have to resort to dedicated LPV-identification schemes that suitably reveal the parameter dependence in the system under investigation. Also further research should be directed toward finding systematic procedures to choose the basis functions that mimic the nonlinear bahavior of the plant in the controller for exploitation in techniques such as the one developed here. Finally, the main drawback of the proposed method is that it is computationally very demanding. The performance of the current LMI solvers limits practical design to systems with no more than 20 states and two or three varying parameters.

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References

[1]Shamma, J. S., and Athans, M., "Gain-Scheduling: Potential Hazards and Possible Remedies," *IEEE Control Systems Magazine*, Vol. 13, June 1992, pp. 101–107.

[2]Becker, G., "Quadratic Stability and Performance of Linear Parameter Dependent Systems," Ph.D. Thesis, Dept. of Mechanical Engineering, Univ. of California, Berkeley, CA, 1993.

[3]Apkarian, P., and Gahinet, P., "A Convex Characterization of Gain-Scheduled H_∞-Controllers," *IEEE Transactions on Automatic Control*, Vol. 40, May 1995, pp. 853–864.

[4]Helmersson, A., "Methods for Robust Gain-Scheduling," Ph.D. Thesis, Div. of Automatic Control, Linköping Univ., Sweden, 1995.

[5]Packard, A., "Gain Scheduling via Linear Fractional Transformations," *Systems and Control Letters*, Vol. 22, Oct. 1994, pp. 79–92.

[6]Scherer, C. W., "Mixed H_2/H_∞ Control," *Trends in Control, A European Perspective*, edited by A. Isidori, Springer–Verlag, Berlin, 1995, pp. 173–216.

[7]Apkarian, P., and Adams, R. J., "Advanced Gain-Scheduling Techniques for Uncertain Systems, *IEEE Transactions on Control Systems Technology*, (to be published).

[8]Becker, G., "Additional Results on Parameter Dependent Controllers for LPV Systems," *IFAC 13th Triennial World Congress* (San Francisco, CA), Vol. G, Inst. of Electrical and Electronics Engineers, New York, 1996, pp. 351–356.

[9]Lee, L. H., and Spillman, M., "Robust, Reduced-Order, Linear Parameter-Varying Flight Control for an F-16," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (New Orleans, LA), AIAA, Reston, VA, 1997.

[10]Lee, L. H., "Identification and Robust Control of Linear Parameter-Varying Systems," Ph.D. Thesis, Dept. of Mechanical Engineering, Univ. of California, Berkeley, CA, 1997.

[11]Willemsen, D. M. C., "Linear Matrix Inequalities in Robust Control," M.S. Thesis, Dept. of Aerospace Engineering, Delft Univ. of Technology, Delft, The Netherlands, 1996.

[12]Wu, F., "Control of Linear Parameter Varying Systems," Ph.D. Thesis, Dept. of Mechanical Engineering, Univ. of California, Berkeley, CA, 1995.

[13]Becker, G., and Packard, A., "Robust Performance of Linear Parameterically Varying Systems Using Parameterically-Dependent Linear Feedback," *Systems and Control Letters*, Vol. 23, Feb. 1994, pp. 205–215.

[14]Rantzer, A., and Megretski, A., "System Analysis via Integral Quadratic Constraints," *Proceedings of IEEE Conference on Decision and Control*, Inst. of Electrical and Electronics Engineers, New York, 1994, pp. 3062–3067.

[15]Scorletti, G., and El Ghaoui, L., "Improved Linear Matrix Inequality Conditions for Gain-Scheduling," *Proceedings of the 34th IEEE Conference on Decision and Control*, Inst. of Electrical and Electronics Engineers, New York, 1995, pp. 3626–3631.

[16]Masubuchi, I., Suda, N., and Ohara, A., "LMI-Based Controller Synthesis: A Unified Formulation and Solution," American Control Conf., 1995, pp. 3473–3477.

[17]Scherer, C. W., Gahinet, P., and Chilali, M., "Multi-Objective Output-Feedback Control via LMI Optimization," *IEEE Transactions on Automatic Control*, Vol. 42, pp. 896–911, 1997.

[18]Gahinet, P., Nemirovski, A., Laub, A., and Chilali, M., "LMI Control Toolbox," MathWorks, Inc., Natick, MA, 1994.

[19]Rugh, W. J., Nichols, R. A., and Reichert, R. T., "Gain Scheduling for H_∞ Controllers: A Flight Control Example," *IEEE Transactions on Control Systems Technology*, Vol. 1, Jan. 1993, pp. 69–78.

[20]Packard, A., and Balas, G. J., "Time-Varying Controllers for Missile Autopilots," *1st IEEE Conference on Control Applications* (Dayton, OH), Inst. of Electrical and Electronics Engineers, New York, 1992.

[21]Balas, G. J., Doyle, J. C., Glover, K., Packard, A., and Smith R., "μ-Analysis and Synthesis TOOLBOX, For Use with MATLAB," MathWorks, Inc., Natick, MA, 1993.

[22]Bianic, J. M., Apkarian, P., and Garrard, W. L., "Parameter Varying Control of a High Performance Aircraft," *IFAC 13th Triennial World Congress* (San Fransisco, CA), Inst. of Electrical and Electronics Engineers, New York, 1996.

[23]Scherer, C. W., Njio, R. G. E., and Bennani, S., "Parametrically Varying Flight Control System Design with Full Block Scalings," Control and Decision Conf., San Diego, CA, Dec. 1997.